

CONDITIONAL EXPECTATIONS OF RANDOM HOLOMORPHIC FIELDS ON RIEMANN SURFACES

RENJIE FENG

ABSTRACT. We study two conditional expectations: $K_n(z|p)$ of the expected density of critical points of Gaussian random holomorphic sections $s_n \in H^0(M, L^n)$ of powers of a positive holomorphic line bundle (L, h) over Riemann surfaces (M, ω) given that the random sections vanish at a point and $D_n(z|q)$ of the expected density of zeros given that the random sections has a fixed critical point. The critical points are points $\nabla_{h^n} s_n = 0$ where ∇_{h^n} is the smooth Chern connection of the Hermitian metric h^n . The main result is that the rescaling conditional expectations $K_n(p + \frac{u}{\sqrt{n}}|p)$ and $D_n(q + \frac{u}{\sqrt{n}}|q)$ have universal limits $K_\infty(u|0)$ and $D_\infty(v|0)$ as the power of the line bundle tends to infinity. We will see that the short distance behaviors of these two conditional expectations are quite different: the behavior between critical points and the conditioning zero is neutral while there is a repulsion between zeros and the conditioning critical point. But the long distance behaviors of these two rescaling densities are the same.

1. INTRODUCTION

For Gaussian random polynomials of degree n , we study the conditional expectation of critical points given that the polynomials vanish at a point and the conditional expectation of zeros given that the polynomials have a fixed critical point. More generally, we consider the conditional distribution $K_n(z|p)$ of critical points of Gaussian random holomorphic sections $s_n \in H^0(M, L^n)$ of powers of a positive holomorphic line bundle (L, h) over Riemann surface (M, ω) given that the random sections vanish at a point p and the conditional expectation $D_n(z|q)$ of zeros given that the Chern connection of the random sections vanish at a point q . We will apply a Kac-Rice type formula to derive $K_n(z|p)$ and the probabilistic Poincaré-Lelong formula to derive $D_n(z|q)$, then we rescale them to prove that $K_n(p + \frac{u}{\sqrt{n}}|p)$ and $D_n(q + \frac{u}{\sqrt{n}}|q)$ have universal limits as n tends to infinity.

The motivation of this paper is to study the local behavior between the critical points and zeros of random holomorphic fields. The famous Gauss-Lucas Theorem states that the holomorphic critical points of any polynomial are contained in the convex hull of its zeros. This implies that some non-trivial correlations between zeros and critical points of random polynomials may exist. This problem has been studied recently in [9, 8] for Gaussian random $SU(2)$ polynomials where a two-point correlation function between zeros and critical points is derived. It is also proved that on the $n^{-\frac{1}{2}}$ length-scaled, zeros and critical points appear in rigid pairs. It seems that the similar results hold for holomorphic sections of line bundles over Riemann surfaces. In this article, we study the analogous problems. Instead of two-point correlation between zeros and critical points, we study the conditional

Date: November 10, 2015.

expectations of critical points and zeros. Our essential setting on Riemann surfaces is that the critical points are defined as zeros of the derivative of the smooth Chern connection ∇_{h^n} instead of the meromorphic connection (or locally, the holomorphic derivative $\frac{\partial}{\partial z}$ on a coordinate patch \mathbb{C} of Riemann surfaces).

1.1. Results on critical points. To state our results, we need to recall the definition of Gaussian random holomorphic sections of a line bundle (see §2). We let $(L, h) \rightarrow (M, \omega)$ be a positive Hermitian holomorphic line bundle over a Riemann surface with the Kähler form $\omega = \frac{\sqrt{-1}}{2}\Theta_h$, where Θ_h is the curvature of h . We denote $H^0(M, L^n)$ as the space of global holomorphic sections of n -th tensor power of L . A special case is when $M = \mathbb{CP}^1$ and $L = \mathcal{O}(1)$ the hyperplane line bundle, $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ is the space of homogeneous polynomials of degree n . The Hermitian metric h will induce an inner product on $H^0(M, L^n)$ and thus induces a Gaussian measure $d\gamma_{d_n}$ on $H^0(M, L^n)$, where d_n is the dimension of $H^0(M, L^n)$.

We define $K_n(z|p)$ the conditional expectation of critical points as a $(1, 1)$ -current

$$(1) \quad \int_M \psi K_n(z|p) = \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})} \left(\sum_{z: \nabla_{h^n} s_n = 0} \psi(z) | s_n(p) = 0 \right),$$

for any test function $\psi \in C_0^\infty(M)$ where ∇_{h^n} is the Chern connection. In §3, we will rewrite the right hand side as an expectation taken in the probability space $(H_p^0(M, L^n), d\gamma_{d_n-1}^p)$ with respect to the conditional Gaussian measure $d\gamma_{d_n-1}^p$ (see §3).

In order to get the conditional distribution of the critical points, we need to apply the generalized Kac-Rice formula for complex manifolds [3, 5, 6]. Our first result is the following Kac-Rice type formula for the global $(1, 1)$ -current of $K_n(z|p)$.

Theorem 1. *Let $(L, h) \rightarrow (M, \omega)$ be a positive Hermitian holomorphic line bundle over a compact Riemann surface with the Kähler form $\omega = \frac{\sqrt{-1}}{2}\Theta_h$, let $(H^0(M, L^n), d\gamma_{d_n})$ be a complex Gaussian ensemble defined in §2.2. Then the conditional expectation of the empirical measure of critical points given that the random sections vanish at p is*

$$K_n(z|p) = \left(\int_{\mathbb{C}^2} \frac{1}{\pi^3} \frac{1}{A_n \det \Lambda_n} \exp \left\{ - \left\langle \begin{pmatrix} \xi \\ y \end{pmatrix}, \Lambda_n^{-1} \begin{pmatrix} \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\} ||\xi|^2 - n^2|y|^2| d\ell_y d\ell_\xi \right) \omega(z),$$

where $d\ell_y$ and $d\ell_\xi$ are Lebesgue measures on \mathbb{C} and

$$\Lambda_n = C_n - A_n^{-1} B_n^* B_n$$

where

$$A_n = \partial_z \partial_{\bar{w}} \Pi_n^p(z, w)|_{z=w},$$

$$B_n = (\partial_z \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w}, \partial_z \Pi_n^p(z, w)|_{z=w}),$$

and

$$C_n = \begin{pmatrix} \partial_z^2 \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w} & \partial_z^2 \Pi_n^p(z, w)|_{z=w} \\ \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w} & \Pi_n^p(z, z) \end{pmatrix},$$

where

$$\Pi_n^p(z, w) = \Pi_n(z, w) - \frac{\Pi_n(z, p) \overline{\Pi_n(w, p)}}{\Pi_n(p, p)},$$

where $\Pi_n(z, w)$ is the Bergman kernel which is the projection of the L^2 integral sections to the holomorphic sections (see §4.3).

We rescale the global expression of $K_n(z|p)$ to get the following local behavior between critical points and the conditioning zero,

Theorem 2. *The rescaling limit of the $(1, 1)$ -current of the conditional expectation has a pointwise universal limit,*

$$K_\infty(u|0) := \lim_{n \rightarrow \infty} K_n(p + \frac{u}{\sqrt{n}}|p) = \frac{1}{\pi a_\infty^2} \frac{(\lambda_1^\infty)^2 + (\lambda_2^\infty)^2}{|\lambda_1^\infty| + |\lambda_2^\infty|} \frac{\sqrt{-1}}{2} du \wedge d\bar{u}$$

where

$$a_\infty = 1 + |u|^2, \quad \lambda_1^\infty = 2 + 2|u|^2 + |u|^4, \quad \lambda_2^\infty = -1 + |u|^2 e^{-|u|^2} + e^{-|u|^2}.$$

We will first prove this result for the special case of Gaussian random $SU(2)$ polynomials in §5 and then prove the general cases in §6. To prove this result, we need the estimates of the rescaling limits of the Bergman kernel $\Pi_n(p + \frac{z}{\sqrt{n}}, p + \frac{w}{\sqrt{n}})$ and its derivatives up to order 4.

1.2. Results on zeros. The conditional expectation $D_n(z|q)$ of zeros of Gaussian random holomorphic sections with a fixed critical point is defined similarly,

$$(2) \quad \int_M \psi D_n(z|q) = \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})} \left(\sum_{z: s_n=0} \psi(z) |\nabla_{h^n} s_n(q)|^2 = 0 \right)$$

for any test function ψ .

In §7, we will apply the probabilistic Poincaré-Lelong formula to get,

Theorem 3. *With the same assumptions in Theorem 1, the conditional expectation of the empirical measure of zeros given that the random sections have a critical point at q is*

$$D_n(z|q) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\Pi_n^q(z, z)|,$$

where

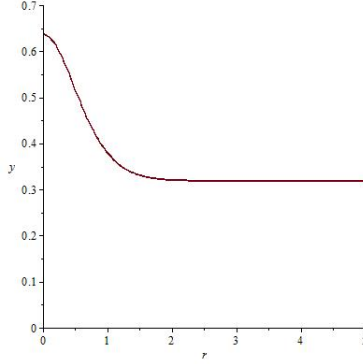
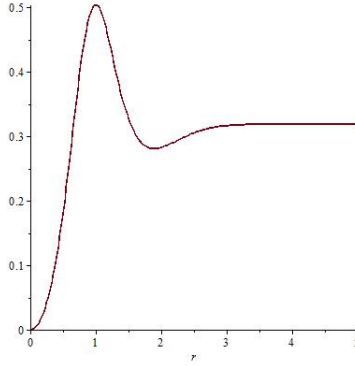
$$\Pi_n^q(z, z) = \Pi_n(z, z) - \frac{|\partial_{\bar{w}} \Pi_n(z, w)|_{w=q}|^2}{(\partial_z \partial_{\bar{z}} \Pi_n)(q, q)}.$$

Furthermore, $D_n(z|q)$ admits the following universal limit,

Theorem 4.

$$D_\infty(v|0) := \lim_{n \rightarrow \infty} D_n(q + \frac{v}{\sqrt{n}}|q) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(e^{|v|^2} - |v|^2 \right).$$

Theorem 2 and Theorem 4 indicate that these two universal limits depend only on the distance $r = |u|$ or $|v|$ between the scaled point and the conditioning fixed point. The following graphs illustrate the growth of the density functions $K_\infty(u|0)$ and $D_\infty(v|0)$ (by discarding the Lebesgue measure).

Graph of $K_\infty(u|0)$ Graph of $D_\infty(v|0)$

Theorem 2 and Theorem 4 also determine some local behaviors between critical points and zeros. It's very surprising to see that the short distance behaviors of $K_\infty(u|0)$ and $D_\infty(v|0)$ are quite different,

$$(3) \quad \lim_{|u| \rightarrow 0} K_\infty(u|0) = \frac{2}{\pi}, \quad \lim_{|v| \rightarrow 0} D_\infty(v|0) = 0.$$

But the long distance behaviors are the same,

$$(4) \quad \lim_{|u| \rightarrow \infty} K_\infty(u|0) = \frac{1}{\pi}, \quad \lim_{|v| \rightarrow \infty} D_\infty(v|0) = \frac{1}{\pi}.$$

Intuitively, the rescaling limit measures the asymptotic probability of finding critical points/zeros in the geodesic ball of length scale $n^{-\frac{1}{2}}$ centered at the conditioning point. Roughly speaking, the limit $D_\infty(v|0)$ tends to 0 as $|v| \rightarrow 0$ indicates that given a critical point at q , it's unlikely to find a zero near q , i.e., there is a 'repulsion' between zeros and the conditioning critical point. Such behavior is quite different from that of the critical points given a zero: the limit of $K_\infty(u|0)$ tends to a constant as $|u| \rightarrow 0$ indicates that critical points and the conditioning zero behave 'neutrally' for the short distance. It should be very interesting to see the behaviors for the higher dimensions.

As a remark, the universal rescaling limits $K_\infty(u|0)$ and $D_\infty(v|0)$ are the rescaling conditional densities for the Bargmann-Fock case with the conditioning point at $z = 0$, i.e., the corresponding conditional densities for the random holomorphic

functions,

$$f(z) = \sum_{j=0}^{\infty} \frac{a_j}{\sqrt{j!}} z^j,$$

where a_j are i.i.d. standard complex Gaussian random variables with mean 0 and variance 1. The reason of this can be tell from the proof: the covariance kernel of Gaussian random holomorphic sections is expressed by the Bergman kernel, while the rescaling limit of the Bergman kernel on any polarized line bundle over Kähler manifolds is universal which is the rescaling limit of the Bergman kernel of the Bargmann-Fock space (see Remark 2).

Acknowledgement: The author would like to thank Steve Zelditch for his generous support for so many years. He would like to thank Bernard Shiffman, Zuoqin Wang, Zhiqin Lu and Zhenan Wang for many helpful discussions. Many thanks also go to Richard Wentworth, Robert Adler and Bo Guan.

2. BACKGROUND

In this section, we will review some basic concepts and notations on Gaussian random holomorphic sections of positive holomorphic line bundles over Riemann surfaces. We refer to [3, 7] for more details.

2.1. Kähler manifolds. Let (M, ω) be a compact Riemann surface (which is a Kähler manifold) with the Kähler form

$$(5) \quad \omega = \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z},$$

where ϕ is the local Kähler potential in a local coordinate patch $U \subset M$. Let $(L, h) \rightarrow (M, h)$ be a positive holomorphic line bundle such that the curvature of the Hermitian metric h

$$(6) \quad \Theta_h = -\frac{\partial^2 \log h}{\partial z \partial \bar{z}} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$$

is a positive $(1, 1)$ form [7]. Let e be a local non-vanishing holomorphic section of L over $U \subset M$ such that locally $L|_U \cong U \times \mathbb{C}$ and the pointwise h -norm of e is $|e|_h = h(e, e)^{1/2}$. Throughout the article, we further assume that the line bundle is polarized, i.e., $\Theta_h = \omega$ or equivalently $|e|_h^2 = h(e, e) = e^{-\phi}$.

We denote by $H^0(M, L^n)$ the space of global holomorphic sections of n -th tensor power of L . Locally, we can write the global holomorphic section of L^n as $s_n = f_n e^{\otimes n}$ where f_n is a holomorphic function on U . We denote the dimension of $H^0(M, L^n)$ by d_n . The Hermitian metric h induces a Hermitian metric h^n on L^n given by $|e^{\otimes n}|_{h^n} = |e|_h^n$, i.e., $|s_n|_{h^n}^2 = |f_n|^2 h^n(e^{\otimes n}, e^{\otimes n}) = |f_n|^2 e^{-n\phi}$.

We decompose the smooth Chern connection $\nabla_{h^n} = \nabla'_{h^n} + \nabla''_{h^n}$ of the Hermitian line bundle (L^n, h^n) into holomorphic and antiholomorphic parts where in the local coordinate $\nabla'_{h^n} = d_z + n\partial \log h$ and $\nabla''_{h^n} = d_{\bar{z}}$. For the polarized line bundles, given a global holomorphic section $s_n = f_n e^{\otimes n}$, we have the following formula [7]

$$(7) \quad \nabla_{h^n} s_n = \left(\frac{\partial f_n}{\partial z} - n \frac{\partial \phi}{\partial z} f_n \right) e^{\otimes n} \otimes dz.$$

We can define an inner product on $H^0(M, L^n)$ as

$$(8) \quad \langle s_{n,1}, s_{n,2} \rangle_{h^n} = \int_M h^n(s_{n,1}, s_{n,2}) \omega.$$

Under the local coordinate it reads,

$$(9) \quad \langle s_{n,1}, s_{n,2} \rangle_{h^n} = \int_M f_{n,1} \overline{f_{n,2}} h^n(e^{\otimes n}, e^{\otimes n}) \omega = \int_M f_{n,1} \overline{f_{n,2}} e^{-n\phi} \omega,$$

where $s_{n,1} = f_{n,1} e^{\otimes n}$ and $s_{n,2} = f_{n,2} e^{\otimes n}$.

2.2. Gaussian random fields. Let's recall that a complex Gaussian measure on \mathbb{C}^k is a measure of the form

$$(10) \quad d\gamma_\Delta = \frac{e^{-\langle \Delta^{-1} z, \bar{z} \rangle}}{\pi^k \det \Delta} d\ell_z^k,$$

where $d\ell_z^k$ denotes Lebesgue measure on \mathbb{C}^k and Δ is a positive definite Hermitian $k \times k$ matrix. The matrix Δ is the covariance matrix.

The inner product (8) induces a complex Gaussian probability measure $d\gamma_{d_n}$ on the space $H^0(M, L^n)$ as,

$$(11) \quad d\gamma_{d_n}(s_n) = \frac{e^{-|a|^2}}{\pi^{d_n}} da, \quad s_n = \sum_{j=1}^{d_n} a_j s_{n,j},$$

where $\{s_{n,1}, \dots, s_{n,d_n}\}$ is an orthonormal basis for $H^0(M, L^n)$ and $\{a_1, \dots, a_{d_n}\}$ are i.i.d. standard complex Gaussian random variables with mean 0 and variance 1.

3. CONDITIONAL EXPECTATION

In this section, we will rewrite the conditional expectations of the empirical measures in (1)(2) by defining the conditional Gaussian measure, then we will derive the covariance kernels for the conditional Gaussian processes.

3.1. Conditional Gaussian measure. Let $d\gamma$ be a complex Gaussian measure on a finite dimensional complex vector space V , let W be a vector subspace of V . We define the conditional Gaussian measure $d\gamma_W$ on W to be the restriction of $d\gamma$ on W associated with the Hermitian inner product on W induced by the inner product on V [14].

This definition can be understood as follows. For simplicity we let W be a vector subspace of V of codimension 1. We let $\{v_1, \dots, v_m\}$ be an orthonormal basis of V and $d\gamma$ be a Gaussian measure induced by the inner product,

$$d\gamma(v) = \frac{e^{-|a|^2}}{\pi^m} da, \quad v = \sum_{j=1}^m a_j v_j,$$

where a_j are i.i.d. standard Gaussian random variables.

Let $\{w_1, \dots, w_{m-1}\}$ be an orthonormal basis of W where the inner product is induced by that of V . Then we extend it to an orthonormal basis $\{w_1, \dots, w_{m-1}, w_m\}$ of V . There exists a unitary group $U = (u_{ij})$ such that $v_j = \sum_{i=1}^m u_{ji} w_i$. Then we can express $v = \sum_{j=1}^m a_j (\sum_{i=1}^m u_{ji} w_i) = \sum_{i=1}^m (\sum_{j=1}^m a_j u_{ji}) w_i$. By the definition of the conditional expectation, the conditional Gaussian measure $d\gamma_W$ induced by $d\gamma$ is actually induced by the conditional Gaussian process

$$w = \sum_{i=1}^{m-1} \left(\sum_{j=1}^m a_j u_{ji} \right) w_i,$$

which is the projection of v to W . The conditional expectation taken with respect to $d\gamma(v)$ is actually taken with respect to $d\gamma_W(w)$. Furthermore, we can compute the covariance kernel of the conditional Gaussian process. The covariance kernel for the Gaussian process v is

$$C_V(v(x), v(y)) = \mathbf{E}_{d\gamma}(v(x)\bar{v}(y)) = \sum_{j=1}^m v_j(x)\bar{v}_j(y) = \sum_{j=1}^m w_j(x)\bar{w}_j(y),$$

and the covariance kernel for the conditional Gaussian process is

$$\begin{aligned} C_W(w(x), w(y)) &= \mathbf{E}_{d\gamma_W}(w(x)\bar{w}(y)) \\ &= \mathbf{E}_{d\gamma} \left(\left[\sum_{i=1}^{m-1} \left(\sum_{j=1}^m a_j u_{ji} \right) w_i \right] \left[\sum_{i=1}^{m-1} \left(\sum_{j=1}^m \bar{a}_j \bar{u}_{ji} \right) \bar{w}_i \right] \right) \\ &= \sum_{j=1}^{m-1} w_j(x)\bar{w}_j(y). \end{aligned}$$

Hence, we have the following crucial relation,

$$(12) \quad C_W(w(x), w(y)) = C_V(v(x), v(y)) - w_m(x)\bar{w}_m(y).$$

3.2. Conditional densities. Now we can define the expected density of critical points of Gaussian random sections given that the random sections vanish at a point p by the conditional Gaussian measure. We need the following bundle-value map

$$(13) \quad T : H^0(M, L^n) \rightarrow L_p^n, \quad s_n \rightarrow s_n(p).$$

We define the kernel of T as $H_p^0(M, L^n) \subset H^0(M, L^n)$ which is a subspace of codimension 1.

We denote C_{s_n} as the empirical measure of critical points of sections,

$$(14) \quad C_{s_n} = \{z \in M : \nabla_{h^n} s_n(z) = 0\}.$$

By definition of conditional Gaussian measure, we have the following relation,

$$(15) \quad \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})}(C_{s_n} | s_n(p) = 0) = \mathbf{E}_{(H_p^0(M, L^n), d\gamma_{d_n-1}^p)}(C_{s_n}),$$

in the sense of distribution,

$$(16) \quad \langle \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})}(C_{s_n} | s_n(p) = 0), \psi \rangle = \mathbf{E}_{(H_p^0(M, L^n), d\gamma_{d_n-1}^p)} \langle C_{s_n}, \psi \rangle,$$

for any test function $\psi \in C_0^\infty(M)$, where $d\gamma_{d_n-1}^p$ is the conditional Gaussian measure which is the restriction of $d\gamma_{d_n}$ on $H_p^0(M, L^n)$.

We denote $K_n(z|p)$ as the conditional expectation $\mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})}(C_{s_n} | s_n(p) = 0)$, then $K_n(z|p)$ is a $(1, 1)$ -current [7] and we can rewrite (16) as

$$(17) \quad \int_M \psi K_n(z|p) = \mathbf{E}_{(H_p^0(M, L^n), d\gamma_{d_n-1}^p)} \left(\sum_{z \in C_{s_n}} \psi(z) \right).$$

We need the following bundle-value linear map to define the conditional expectation of zeros given that the random sections have a critical point at q ,

$$(18) \quad K : H^0(M, L^n) \rightarrow (L^n \otimes T^{*'} M)_q, \quad s_n \rightarrow \nabla_{h^n} s_n(q),$$

where ∇_{h^n} is the Chern connection. We denote the kernel of this linear map as $H_q^0(M, L^n)$. By the Kodaira embedding, $H_q^0(M, L^n)$ is a subspace of $H^0(M, L^n)$ of codimension 1.¹

We denote

$$(19) \quad Z_{s_n} = \{z \in M : s_n(z) = 0\},$$

and denote

$$D_n(z|p) =: \mathbf{E}_{(H^0(M, L^n), d\gamma_{d_n})}(Z_{s_n} | \nabla_{h^n} s_n(p) = 0).$$

Then similarly, we have

$$(20) \quad \int_M \psi D_n(z|q) = \mathbf{E}_{(H_q^0(M, L^n), d\gamma_{d_n-1}^q)} \left(\sum_{z \in Z_{s_n}} \psi(z) \right),$$

where $d\gamma_{d_n-1}^q$ is the conditional Gaussian measure on the subspace $H_q^0(M, L^n)$.

4. PROOF OF THEOREM 1

In this section, we will derive a Kac-Rice type formula for the global expression of the conditional expectation $K_n(z|p)$. The formula may be derived from [1, 3, 5, 6] but we take advantage of some simplifications to speed up the proof.

4.1. Kac-Rice formula. In the local coordinate $U \cong \mathbb{C}$ and a local trivialization of L , we write the conditional Gaussian random sections with a zero at p as $s_n^p = f_n^p e^{\otimes n}$. We will prove the following,

Lemma 1. *The $(1,1)$ -current of the conditional expectation of critical points of sections with a conditioning zero at p with respect to the Gaussian measure $d\gamma_{d_n}$ is*

$$(21) \quad K_n(z|p) = \left(\int_{\mathbb{C}^2} p_z^n(y, 0, \xi) (|\xi|^2 - n^2|y|^2) d\ell_y d\ell_\xi \right) \omega,$$

where $p_z^n(y, s, \xi)$ is the joint density of the conditional Gaussian processes $(f_n^p, \frac{\partial f_n^p}{\partial z}, \frac{\partial^2 f_n^p}{\partial z^2})$; $d\ell_y$ and $d\ell_\xi$ are Lebesgue measures on \mathbb{C} .

Proof. The strategy to get this formula is to find the local expression in a coordinate path $U \cong \mathbb{C}$, then turn it to be global.

We denote

$$(22) \quad \Omega_p = \{z \in \mathbb{C} : (\frac{\partial f_n^p}{\partial z} - n \frac{\partial \phi}{\partial z} f_n^p) e^{-\frac{n\phi}{2}} = 0\},$$

then Ω_p is the same as the set of critical points $\{z \in \mathbb{C} : \nabla_{h^n} s_n^p = 0\}$ which is $\{z \in \mathbb{C} : \frac{\partial f_n^p}{\partial z} - n \frac{\partial \phi}{\partial z} f_n^p = 0\}$ on the local coordinate patch (recall (7)).

We first introduce some notations:

$$(23) \quad p_n = f_n^p e^{-\frac{n\phi}{2}}, \quad q_n = (\frac{\partial f_n^p}{\partial z} - n \frac{\partial \phi}{\partial z} f_n^p) e^{-\frac{n\phi}{2}}, \quad r_n = \frac{\partial^2 f_n^p}{\partial z^2} e^{-\frac{n\phi}{2}},$$

then p_n , q_n and r_n are all complex Gaussian random variables.

¹We thank Prof. Zhiqin Lu for clarifying this.

By definition of the delta function, for any test functions $\psi \in C_0^\infty(\mathbb{C})$ we have,

$$\begin{aligned}
& \langle \sum_{z \in \Omega_p} \delta_z, \psi \rangle \\
&= \sum_{z: q_n(z)=0} \psi(z) \\
&= \int_{\mathbb{C}} \delta_0(q_n) \psi(z) \frac{\sqrt{-1}}{2} dq_n \wedge d\bar{q}_n \\
&= \int_{\mathbb{C}} \delta_0(q_n) \psi(z) \left| \frac{\partial q_n}{\partial z} \right|^2 - \left| \frac{\partial q_n}{\partial \bar{z}} \right|^2 \right| d\ell_z,
\end{aligned}$$

where $d\ell_z$ is the Lebesgue measure on \mathbb{C} .

By direct computations, we have,

$$\begin{aligned}
\frac{\partial q_n}{\partial z} &= (\partial^2 f_n^p - n\partial\phi\partial f_n^p - n\partial^2\phi f_n^p) e^{-\frac{n\phi}{2}} - \frac{n}{2} \partial\phi q_n \\
&= r_n - n\partial\phi q_n - n^2(\partial\phi)^2 p_n - n\partial^2\phi p_n - \frac{n}{2} \partial\phi q_n
\end{aligned}$$

and

$$\frac{\partial q_n}{\partial \bar{z}} = -n\partial\bar{\partial}\phi p_n - \frac{n}{2} \bar{\partial}\phi q_n.$$

By taking expectation on both sides, we have locally,

$$\begin{aligned}
& \mathbf{E} \langle \sum_{z \in \Omega_p} \delta_z, \psi \rangle \\
&= \mathbf{E} \int_{\mathbb{C}} \delta_0(q_n) \psi(z) \left| \frac{\partial q_n}{\partial z} \right|^2 - \left| \frac{\partial q_n}{\partial \bar{z}} \right|^2 \right| d\ell_z \\
&= \int_{\mathbb{C}^3} \psi(z) p_z^n(y, 0, \xi) \left| |\xi - n^2(\partial\phi)^2 y - n\partial^2\phi y|^2 - n^2|\partial\bar{\partial}\phi|^2 |y|^2 \right| d\ell_\xi d\ell_y d\ell_z
\end{aligned}$$

where $p_z^n(y, s, \xi)$ is the joint probability of the Gaussian random field (p_n, q_n, r_n) , $d\ell_\xi$ and $d\ell_y$ are Lebesgue measures on \mathbb{C} . Thus the conditional density is locally given by the (1,1)-current,

$$(24) \quad \left(\int_{\mathbb{C}^2} p_z^n(y, 0, \xi) \left| |\xi - n^2(\partial\phi)^2 y - n\partial^2\phi y|^2 - n^2|\partial\bar{\partial}\phi|^2 |y|^2 \right| d\ell_\xi d\ell_y \right) d\ell_z$$

on the local coordinate patch $U \cong \mathbb{C}$ of Riemann surfaces.

Now we need to get the global expression for the conditional density. Since the conditional expectation is a (1,1)-current globally defined on the Riemann surface (which is also independent of the local coordinate and the local frame), it's sufficient to find the formula when we freeze at a point and the formula will turn out to be global. For this purpose, given a complex m -dimensional Kähler manifold $(L, h) \rightarrow (M, \omega)$, we freeze at a point z_0 as the origin of the coordinate patch and choose the Kähler normal coordinate $\{z_j\}$ as well as an adapted frame e_L of the line bundle L around z_0 . It is well-known that in terms of Kähler normal coordinates $\{z_j\}$, the Kähler potential ϕ has the following expansion in the neighborhood of the origin z_0 ,

$$(25) \quad \phi(z, \bar{z}) = \|z\|^2 - \frac{1}{4} \sum R_{j\bar{k}p\bar{q}}(z_0) z_j \bar{z}_{\bar{k}} z_p \bar{z}_{\bar{q}} + O(\|z\|^5).$$

And thus,

$$(26) \quad \phi(z_0) = 0, \partial\phi(z_0) = 0, \partial^2\phi(z_0) = 0, \partial\bar{\partial}\phi(z_0) = 1, \omega(z_0) = d\ell_z.$$

An example on the Kähler normal coordinate and the adapted frame is present in §5, it is the affine coordinate for the Fubini-Study metric of the hyperplane line bundle over the complex projective space $(\mathcal{O}(1), h_{FS}) \rightarrow (\mathbb{CP}^1, \omega_{FS})$. We also refer to §3.1 in [5] for more details.

After we choose the normal coordinate at z_0 , by identities (26), the joint density of the Gaussian processes (p_n, q_n, r_n) (recall (23)) at z_0 should be the same as the joint density of Gaussian processes $(f_n^p, \partial f_n^p, \partial^2 f_n^p)$. Hence, by (26) again, the local expression (24) admits the following global expression,

$$\mathbf{E} \left(\sum_{z \in \Omega_p} \delta_z \right) = \left(\int_{\mathbb{C}^2} p_z^n(y, 0, \xi) (|\xi|^2 - n^2|y|^2) d\ell_y d\ell_\xi \right) \omega := K_n(z|p),$$

where $p_z^n(y, s, \xi)$ is the joint density of the Gaussian processes $(f_n^p, \frac{\partial f_n^p}{\partial z}, \frac{\partial^2 f_n^p}{\partial z^2})$. This completes the proof of Lemma 1. \square

4.2. Proof of Theorem 1. In this subsection, we will calculate the joint density p_z^n of the Gaussian processes of $(f_n^p, \partial f_n^p, \partial^2 f_n^p)$.

For the conditional Gaussian processes $(f_n^p, \partial f_n^p, \partial^2 f_n^p)$, the joint density is given by the formula [1]

$$(27) \quad p_z^n(y, s, \xi) = \frac{1}{\pi^3} \frac{1}{\det \Delta_n} \exp \left\{ \left\langle \begin{pmatrix} y \\ s \\ \xi \end{pmatrix}, (\Delta_n)^{-1} \begin{pmatrix} \bar{y} \\ \bar{s} \\ \bar{\xi} \end{pmatrix} \right\rangle \right\},$$

where Δ_n is the covariance matrix of the conditional Gaussian process $(f_n^p, \partial f_n^p, \partial^2 f_n^p)$.

We rearrange the order of the Gaussian processes and write $\tilde{\Delta}_n$ as the covariance matrix of $(\partial f_n^p, \partial^2 f_n^p, f_n^p)$, then we rewrite

$$(28) \quad p_z^n(y, s, \xi) = \frac{1}{\pi^3} \frac{1}{\det \tilde{\Delta}_n} \exp \left\{ \left\langle \begin{pmatrix} s \\ \xi \\ y \end{pmatrix}, (\tilde{\Delta}_n)^{-1} \begin{pmatrix} \bar{s} \\ \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\}.$$

The covariance matrix is then given by

$$(29) \quad \tilde{\Delta}_n = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix}_{3 \times 3},$$

where

$$A_n = \partial_z \partial_{\bar{w}} \Pi_n^p(z, w)|_{z=w},$$

$$B_n = (\partial_z \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w}, \partial_z \Pi_n^p(z, w)|_{z=w}),$$

and

$$C_n = \begin{pmatrix} \partial_z^2 \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w} & \partial_z^2 \Pi_n^p(z, w)|_{z=w} \\ \partial_{\bar{w}}^2 \Pi_n^p(z, w)|_{z=w} & \Pi_n^p(z, z) \end{pmatrix},$$

where $\Pi_n^p(z, w)$ is the covariance kernel of the conditional Gaussian process f_n^p .

Thus, when $s = 0$, by some element matrix computations, we have,

Lemma 2. *With all notations above,*

$$(30) \quad p_z^n(y, 0, \xi) = \frac{1}{\pi^3} \frac{1}{A_n \det \Lambda_n} \exp \left\{ - \left\langle \begin{pmatrix} \xi \\ y \end{pmatrix}, \Lambda_n^{-1} \begin{pmatrix} \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\},$$

where

$$(31) \quad \Lambda_n = C_n - A_n^{-1} B_n^* B_n.$$

We combine Lemma 1 and Lemma 2 to rewrite,

$$K_n(z|p) = \left(\int_{\mathbb{C}^2} \frac{1}{\pi^3} \frac{1}{A_n \det \Lambda_n} \exp \left\{ - \left\langle \begin{pmatrix} \xi \\ y \end{pmatrix}, \Lambda_n^{-1} \begin{pmatrix} \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\} ||\xi|^2 - n^2|y|^2| d\ell_y d\ell_\xi \right) \omega.$$

Hence, we will complete the proof of Theorem 1 once we derive the expression of the covariance kernel $\Pi_n^p(z, w)$ of the conditional Gaussian process. This is computed in the next subsection.

4.3. Covariance kernel. The Bergman kernel is the orthogonal projection from the L^2 integral sections to the holomorphic sections,

$$(32) \quad \Pi_n(z, w) : L^2(M, L^n) \rightarrow H^0(M, L^n)$$

with respect to the inner product (8). It has the following reproducing property

$$(33) \quad \langle s_n(z), \Pi_n(z, w) \rangle_{h^n} = s_n(w),$$

where $s_n \in H^0(M, L^n)$ is a global holomorphic section. Let $\{s_{n,1}, \dots, s_{n,d_n}\}$ be any orthonormal basis of $H^0(M, L^n)$ with respect to the inner product (8), then we have,

$$(34) \quad \Pi_n(z, w) = \sum_{j=1}^{d_n} s_{n,j}(z) \otimes \overline{s_{n,j}(w)}.$$

It's easy to check that $\Pi_n(z, w)$ is also the covariance kernel of the Gaussian process $(H^0(M, L^n), d\gamma_{d_n})$ defined by (11).

Recall $H_0^p \subset H^0(M, L^n)$ is the space of holomorphic sections vanishing at p . Let $\{s_{n,1}^p, \dots, s_{n,d_n-1}^p\}$ be an orthonormal basis of H_0^p with respect to the inner product (8). By the reproducing property of the Bergman kernel $\Pi_n(z, w)$, one can show that the holomorphic sections $\{s_{n,1}^p, \dots, s_{n,d_n-1}^p, \Phi_n^p\}$ will be an orthonormal basis for $H^0(M, L^n)$ (see equation (3.7) in [14] for more details) where (by discarding the local frame $e^{\otimes n}$)

$$(35) \quad \Phi_n^p(z) = \frac{\Pi_n(z, p)}{\sqrt{\Pi_n(p, p)}}.$$

Recall relation (12), then the covariance kernel of the conditional Gaussian process is

$$(36) \quad \Pi_n^p(z, w) = \Pi_n(z, w) - \Phi_n^p(z) \overline{\Phi_n^p(w)}.$$

Hence, we complete the proof of Theorem 1.

4.4. Further simplification. We can further simplify the expression of $K_n(z|p)$ in Theorem 1 as follows. Let $H = (\xi, y)$, then we can rewrite

$$\int_{\mathbb{C}^2} \frac{1}{\pi^3} \frac{1}{A_n \det \Lambda_n} \exp \left\{ - \left\langle \begin{pmatrix} \xi \\ y \end{pmatrix}, \Lambda_n^{-1} \begin{pmatrix} \bar{\xi} \\ \bar{y} \end{pmatrix} \right\rangle \right\} ||\xi|^2 - n^2|y|^2| d\ell_y d\ell_\xi$$

as

$$\frac{1}{\pi^3} \frac{1}{A_n \det \Lambda_n} \int_{\mathbb{C}^2} e^{-H \Lambda_n^{-1} H^*} |H Q_n H^*| d\ell_H,$$

where $Q_n = \begin{pmatrix} 1 & 0 \\ 0 & -n^2 \end{pmatrix}$ and $d\ell_H$ is the Lebesgue measure on \mathbb{C}^2 .

We change variable $H \rightarrow H\Lambda_n^{-\frac{1}{2}}$ to get

$$\frac{1}{\pi^3} \frac{1}{A_n} \int_{\mathbb{C}^2} e^{-HH^*} |H\Lambda_n^{\frac{1}{2}} Q_n \Lambda_n^{\frac{1}{2}} H^*| d\ell_H.$$

We diagonalize $\Lambda_n^{\frac{1}{2}} Q_n \Lambda_n^{\frac{1}{2}}$ with eigenvalues λ_1^n, λ_2^n (note that it is easy to check λ_1^n and λ_2^n are also eigenvalues of $\Lambda_n Q_n$) to simplify the above integration as

$$\begin{aligned} & \frac{1}{\pi^3} \frac{1}{A_n} \int_{\mathbb{C}^2} |\lambda_1^n| |y|^2 + \lambda_2^n |\xi|^2 |e^{-|y|^2 - |\xi|^2}| d\ell_y d\ell_\xi \\ &= \frac{1}{\pi} \frac{1}{A_n} \int_0^\infty \int_0^\infty |\lambda_1^n x + \lambda_2^n y| e^{-x-y} dx dy \\ &= \frac{1}{\pi} \frac{1}{A_n} \frac{(\lambda_1^n)^2 + (\lambda_2^n)^2}{|\lambda_1^n| + |\lambda_2^n|}. \end{aligned}$$

Thus we have,

Lemma 3. *The conditional expectation is*

$$(37) \quad K_n(z|p) = \frac{1}{\pi} \frac{1}{A_n} \frac{(\lambda_1^n)^2 + (\lambda_2^n)^2}{|\lambda_1^n| + |\lambda_2^n|} \omega(z),$$

where λ_1^n and λ_2^n are eigenvalues of $(\Lambda_n Q_n)(z)$.

5. CALCULATIONS OF THEOREM 2 FOR GAUSSIAN RANDOM $SU(2)$ POLYNOMIALS

In this section, we will derive the rescaling conditional density of critical points for Gaussian random $SU(2)$ polynomials. This is the case where $M = \mathbb{CP}^1 \cong S^2$ and L is the hyperplane line bundle $\mathcal{O}(1)$. The global holomorphic sections of $\mathcal{O}(1)$ are linear functions on \mathbb{C}^2 and hence the global holomorphic sections of $L^n = \mathcal{O}(n)$ are homogeneous polynomials of degree n .

The Kähler form on \mathbb{CP}^1 is the Fubini-Study form. In an affine coordinate, the Kähler form and the Kähler potential for the Fubini-Study metric are

$$(38) \quad \omega_{FS} = \frac{\sqrt{-1}}{2} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}, \quad \phi(z) = \log(1 + |z|^2).$$

It's easy to check that ϕ satisfies (26) and the affine coordinate is actually the Kähler normal coordinate at $z_0 = 0$.

We equip $\mathcal{O}(1)$ with its Fubini-Study metric. In fact, we can choose an adapted frame $e(z)$ such that

$$|e(z)|_{h_{FS}}^2 = e^{-\phi} = \frac{1}{1 + |z|^2}.$$

Hence, an orthonormal basis of $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ under the inner product (8) is given by

$$\left\{ \left(\sqrt{(n+1) \binom{n}{j}} z^j \right) e^{\otimes n} \right\}_{j=0}^n.$$

Throughout the article, we will discard the local frame e^{\otimes} for simplicity.

The Gaussian linear combination of the above basis is Gaussian random $SU(2)$ polynomials and the distribution of zeros of such polynomials is invariant under the rotation of S^2 [10].

By (34), the Bergman kernel for the Fubini-Study case is

$$\Pi_n^{SU(2)}(z, w) = (n+1)(1+z\bar{w})^n.$$

By the expression of $K_n(z|p)$ in Theorem 1, the expected density of critical points is unchanged when the covariance kernel is multiplied by a constant (or equivalently the Gaussian process is multiplied by a constant). In the following computations, for simplicity, we can replace the Bergman kernel $\Pi_n^{SU(2)}(z, w)$ by the normalized Bergman kernel

$$\Pi_n(z, w) = (1+z\bar{w})^n.$$

By formula (36), we have the following expression for the covariance kernel of the conditional Gaussian measure

$$\Pi_n^p(z, w) = (1+z\bar{w})^n - \frac{(1+z\bar{p})^n(1+p\bar{w})^n}{(1+p\bar{p})^n}.$$

Now let's compute the matrices B_n and C_n for $H^0(\mathbb{CP}^1, \mathcal{O}(n))$. Indeed, we have,

$$\frac{\partial \Pi_n^p}{\partial z} = n(1+z\bar{w})^{n-1}\bar{w} - \frac{n\bar{p}(1+z\bar{p})^{n-1}(1+p\bar{w})^n}{(1+p\bar{p})^n}$$

$$\frac{\partial^2 \Pi_n^p}{\partial z \partial \bar{w}} = n(1+z\bar{w})^{n-1} + zn(n-1)(1+z\bar{w})^{n-2}\bar{w} - \frac{n^2 p \bar{p} (1+z\bar{p})^{n-1}(1+p\bar{w})^{n-1}}{(1+p\bar{p})^n}$$

$$\frac{\partial^2 \Pi_n^p}{\partial^2 z} = n(n-1)(1+z\bar{w})^{n-2}\bar{w}^2 - \frac{n(n-1)\bar{p}^2(1+z\bar{p})^{n-2}(1+p\bar{w})^n}{(1+p\bar{p})^n}$$

$$\frac{\partial^3 \Pi_n^p}{\partial^2 z \partial \bar{w}} = 2n(n-1)(1+z\bar{w})^{n-2}\bar{w} + n(n-1)(n-2)(1+z\bar{w})^{n-3}z\bar{w}^2 - \frac{n^2(n-1)p\bar{p}^2(1+z\bar{p})^{n-2}(1+p\bar{w})^{n-1}}{(1+p\bar{p})^n}$$

$$\begin{aligned} \frac{\partial^4 \Pi_n^p}{\partial^2 z \partial^2 \bar{w}} &= 2n(n-1)(1+z\bar{w})^{n-2} + 4n(n-1)(n-2)(1+z\bar{w})^{n-3}z\bar{w} + n(n-1)(n-2)(n-3)(1+z\bar{w})^{n-4}z^2\bar{w}^2 \\ &\quad - \frac{n^2(n-1)^2 p^2 \bar{p}^2 (1+z\bar{p})^{n-2}(1+p\bar{w})^{n-2}}{(1+p\bar{p})^n} \end{aligned}$$

Throughout the article, the notation $a_n \sim b_n$ means the asymptotics $a_n = b_n + o(b_n)$ as n large enough; for simplicity we will discard the negligible terms $o(b_n)$ in some steps which do not contribute in the pointwise limit as $n \rightarrow \infty$ in order to keep track of the leading order term, although the precise estimates for all errors terms can be derived.

In order to get the rescaling density $K_n(p + \frac{u}{\sqrt{n}}|p)$ around p , we choose the affine coordinate at $p = 0$ with $z = \frac{u}{\sqrt{n}}$. By Lemma 3, we need to find rescaling limits of $\lambda_1(\frac{u}{\sqrt{n}})$ and $\lambda_2(\frac{u}{\sqrt{n}})$ where λ_1 and λ_2 are eigenvalues of matrix $\Lambda_n Q_n$, or equivalently, we need to find the estimates of two eigenvalues of matrix $(\Lambda_n Q_n)(\frac{u}{\sqrt{n}})$.

We first have the following asymptotics as n large enough,

$$(39) \quad A_n\left(\frac{u}{\sqrt{n}}\right) = n\left(1 + \frac{|u|^2}{n}\right)^{n-1} + (n-1)|u|^2\left(1 + \frac{|u|^2}{n}\right)^{n-2} \sim n(1 + |u|^2)e^{|u|^2}.$$

Similarly, we have,

$$B_n\left(\frac{u}{\sqrt{n}}\right) \sim (2n^{\frac{3}{2}}u + n^{\frac{3}{2}}u|u|^2, n^{\frac{1}{2}}\bar{u})e^{|u|^2},$$

and

$$C_n\left(\frac{u}{\sqrt{n}}\right) \sim e^{|u|^2} \begin{pmatrix} 2n^2 + 4n^2|u|^2 + n^2|u|^4 & n\bar{u}^2 \\ nu^2 & 1 - e^{-|u|^2} \end{pmatrix}.$$

Since $\Lambda_n = C_n - A_n^{-1}B_n^*B_n$, it's easy to compute

$$(\Lambda_n Q_n)\left(\frac{u}{\sqrt{n}}\right) \sim \frac{n^2 e^{|u|^2}}{1 + |u|^2} \times \begin{pmatrix} 2 + 2|u|^2 + |u|^4 & 0 \\ 0 & -1 + e^{-|u|^2} + |u|^2 e^{-|u|^2} \end{pmatrix}.$$

Hence, the eigenvalues of $(\Lambda_n Q_n)(\frac{u}{\sqrt{n}})$ satisfy asymptotics,

$$(40) \quad \lambda_1\left(\frac{u}{\sqrt{n}}\right) \sim \frac{n^2(2 + 2|u|^2 + |u|^4)e^{|u|^2}}{1 + |u|^2} > 0$$

and

$$(41) \quad \lambda_2\left(\frac{u}{\sqrt{n}}\right) \sim \frac{n^2(-1 + e^{-|u|^2} + |u|^2 e^{-|u|^2})e^{|u|^2}}{1 + |u|^2} \leq 0.$$

For the Fubini-Study metric, we have the following estimate,

$$(42) \quad \lim_{n \rightarrow \infty} n\omega_{FS}\left(\frac{u}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} n \frac{\sqrt{-1}}{2} \frac{d\frac{u}{\sqrt{n}} \wedge d\frac{\bar{u}}{\sqrt{n}}}{(1 + |\frac{u}{\sqrt{n}}|^2)^2} = \frac{\sqrt{-1}}{2} du \wedge d\bar{u}.$$

As a remark, this estimate is true for any Kähler metric ω by (26), i.e.,

$$(43) \quad \lim_{n \rightarrow \infty} n\omega(p + \frac{u}{\sqrt{n}}) = \frac{\sqrt{-1}}{2} du \wedge d\bar{u}.$$

If we combine Lemma 3 with asymptotics (39)(40)(41)(42), we have the limit,

$$\lim_{n \rightarrow \infty} K_n(p + \frac{u}{\sqrt{n}}|p) = \frac{1}{\pi a_\infty^2} \frac{(\lambda_1^\infty)^2 + (\lambda_2^\infty)^2}{|\lambda_1^\infty| + |\lambda_2^\infty|} \frac{\sqrt{-1}}{2} du \wedge d\bar{u}$$

with a_∞ , λ_1^∞ and λ_2^∞ given in Theorem 2. Hence we prove Theorem 2 for Gaussian random $SU(2)$ polynomials.

Remark 1. In [14], the authors studied the rescaling limit of the expected density of zeros given that the random sections vanish at a point. The expected (conditional) density of zeros of Gaussian random holomorphic functions can be derived by the probabilistic Poincaré-Lelong formula (see §6). In fact, for Gaussian random $SU(2)$ polynomials, we have the following explicit global expression,

$$\begin{aligned} & \mathbf{E}_{d\gamma_{d_n}}^{SU(2)}\left(\sum_{z: s_n(z)=0} \delta_z |s_n(p) = 0\right) \\ &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \Pi_n^p(z, w) \\ &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left((1 + z\bar{w})^n - \frac{(1 + z\bar{p})^n (1 + p\bar{w})^n}{(1 + p\bar{p})^n} \right). \end{aligned}$$

Hence, the rescaling limit of the above density by choosing the affine coordinate at $p = 0$ with $z = \frac{u}{\sqrt{n}}$ will be

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(e^{|u|^2} - 1),$$

which is one of the main results Corollary 1.3 of [14].

6. PROOF OF THEOREM 2

In this section, we will prove Theorem 2 for any Riemann surfaces.

By Lemma 2, the joint density p_z^n only depends on the Bergman kernel and its derivatives up to order 4; thus the rescaling limit of the conditional expectation should only depend on the rescaling limits of Bergman kernel and its derivatives. We will see that all these rescaling limits are universal.

The Bergman kernel has the following Tian-Yau-Zelditch C^∞ -expansion on the diagonal for Riemann surfaces [12, 15, 16, 17],

$$(44) \quad \Pi_n(z, z) = ne^{n\phi}(1 + a_1(z)n^{-1} + a_2(z)n^{-2} + \dots),$$

where a_1 is the scalar curvature of ω .

Integrating over M with respect to $e^{-n\phi}\omega$ gives the well-known dimension polynomial,

$$(45) \quad d_n = n(1 + n^{-1} \int_M a_1 \omega + n^{-2} \int_M a_2 \omega + \dots).$$

The proof of the full expansion (44) makes use of Boutet de Monvel-Sjostrand parametrix construction [17]. Actually the same construction can be carried out to derive the rescaling limits of the Bergman kernel off diagonal. We also remark that the estimates of the Bergman kernel off diagonal are studied by Dai-Liu-Ma [4].

Let's choose the Kähler normal coordinate at p . First, if we apply identities (26), we have the following on diagonal asymptotics at p ,

$$(46) \quad \Pi_n(p, p) = n(1 + a_1(p)n^{-1} + a_2(p)n^{-2} + \dots).$$

Regarding the rescaling limit of the Bergman kernel, we have the following universal limit,

$$(47) \quad \Pi_n(p + \frac{u}{\sqrt{n}}, p + \frac{v}{\sqrt{n}}) = n(e^{u \cdot \bar{v}} + \frac{1}{\sqrt{n}} p_1 + \dots),$$

where p_1 is a homogeneous polynomial and the error terms have precise estimates (see §5 in [13]).

Remark 2. The term $ne^{u \cdot \bar{v}}$ is actually the rescaling limit of the Bergman kernel for the Bargmann-Fock space. We refer to [3] for more details.

Regarding the rescaling limit of the Bergman kernel and its derivatives on the diagonal, we have the following estimates (we refer [2, 3] for more details),

$$\begin{aligned}
\Pi_n\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= ne^{|u|^2} + O(n^{\frac{1}{2}}), \\
(\partial\Pi_n)\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= n^{\frac{3}{2}}\bar{u}e^{|u|^2} + O(n), \\
(\partial^2\Pi_n)\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= n^2\bar{u}^2e^{|u|^2} + O(n^{\frac{3}{2}}), \\
(\partial\bar{\partial}\Pi_n)\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= n^2e^{|u|^2}(1 + |u|^2) + O(n^{\frac{3}{2}}), \\
(\partial^2\bar{\partial}\Pi_n)\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= n^{\frac{5}{2}}\bar{u}e^{|u|^2}(2 + |u|^2) + O(n^2), \\
(\partial^2\bar{\partial}^2\Pi_n)\left(\frac{u}{\sqrt{n}}, \frac{u}{\sqrt{n}}\right) &= n^3e^{|u|^2}(2 + 4|u|^2 + |u|^4) + O(n^{\frac{5}{2}}).
\end{aligned}$$

And similarly, we have,

$$\Pi_n\left(\frac{u}{\sqrt{n}}, 0\right) = n + O(n^{\frac{1}{2}}), \quad (\partial\Pi_n)\left(\frac{u}{\sqrt{n}}, 0\right) = O(n), \quad (\partial^2\Pi_n)\left(\frac{u}{\sqrt{n}}, 0\right) = O(n^{\frac{3}{2}}).$$

Now we can get the estimates of the covariance matrix,

$$\begin{aligned}
A_n\left(\frac{u}{\sqrt{n}}\right) &= \partial_z \partial_{\bar{w}} \Pi_n^p(z, w)|_{z=w=\frac{u}{\sqrt{n}}, 0} \\
&= \partial_z \partial_{\bar{w}} \Pi_n(z, w)|_{z=w=\frac{u}{\sqrt{n}}} - \frac{\partial_z \Pi(z, p) \overline{\partial_{\bar{w}} \Pi(w, p)}}{\Pi_n(p, p)}|_{z=w=\frac{u}{\sqrt{n}}, p=0} \\
&= n^2 e^{|u|^2} (1 + |u|^2) + O(n^{\frac{3}{2}}) - \frac{O(n^2)}{n(1 + a_1(p)n^{-1} + O(n^{-2}))} \\
&= n^2 (1 + |u|^2) e^{|u|^2} + O(n^{\frac{3}{2}})
\end{aligned}$$

The similar computations yield,

$$B_n\left(\frac{u}{\sqrt{u}}\right) = \left(n^{\frac{5}{2}}u(2 + |u|^2)e^{|u|^2} + O(n^2), n^{\frac{3}{2}}\bar{u}e^{|u|^2} + O(n)\right)$$

and

$$C_n\left(\frac{u}{\sqrt{u}}\right) = \begin{pmatrix} n^3(2 + 4|u|^2 + |u|^4)e^{|u|^2} + O(n^{\frac{5}{2}}) & n^2\bar{u}^2e^{|u|^2} + O(n^{\frac{3}{2}}) \\ n^2u^2e^{|u|^2} + O(n^{\frac{3}{2}}) & n(e^{|u|^2} - 1) + O(n^{\frac{1}{2}}) \end{pmatrix}.$$

Note that the above estimates are the same as the ones in §5 (except an extra factor n since we used normalized Bergman kernel in §5), hence Theorem 2 follows the same computations as in §5 by finding the estimates of the eigenvalues of $(\Lambda_n Q_n)\left(\frac{u}{\sqrt{n}}\right)$.

7. PROOFS OF THEOREM 3

In this section, we will apply the probabilistic Poincaré-Lelong formula to derive a global formula for $D_n(z|q)$ of the empirical measure of zeros with a conditioning critical point.

7.1. Poincaré-Lelong formula. Given a global holomorphic sections s_n of a positive holomorphic line bundle over Kähler manifolds, we denote Z_{s_n} as the empirical measure of zeros of s_n . We write locally $s_n = f_n e^{\otimes n}$, then the classical Poincaré-Lelong formula states that [7]

$$(48) \quad Z_{s_n} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f_n|^2.$$

Taking the expectation on both sides, we have the following probabilistic Poincaré-Lelong formula [11, 14]: Let $S \subset H^0(M, L^n)$ be a Gaussian random field with covariance kernel $\Pi_S(z, w)$, then

$$(49) \quad \mathbf{E}_S(Z_{s_n}) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\Pi_S(z, z)|.$$

7.2. Proof of Theorem 3. Now we turn to prove Theorem 3. Recall (20), we rewrite the conditional expectation of zeros $D_n(z|q)$ as

$$(50) \quad D_n(z|q) = \mathbf{E}_{(H_q^0(M, L^n), d\gamma_{d_n-1}^q)}(Z_{s_n}).$$

By probabilistic Poincaré-Lelong formula (49), we have,

$$(51) \quad D_n(z|q) = \mathbf{E}_{(H_q^0(M, L^n), d\gamma_{d_n-1}^q)}(Z_{s_n}) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\Pi_n^q(z, z)|.$$

where $\Pi_n^q(z, w)$ is the covariance kernel of the conditional Gaussian random sections $(H_q^0(M, L^n), d\gamma_{d_n-1}^q)$, where $H_q^0(M, L^n)$ is the kernel of the linear map $s_n \rightarrow \nabla_{h^n} s_n$ (18).

The Kodaira embedding implies that $H_q^0(M, L^n)$ is a subspace of $H^0(M, L^n)$ of codimension 1. Let $\{s_{n,1}^q, \dots, s_{n,d_n-1}^q\}$ be an orthonormal basis of $H_q^0(M, L^n)$ with respect to the inner product (8). Such basis satisfies $\nabla_{h^n} s_{n,j}^q(q) = 0$ for all $j = 1, \dots, d_n - 1$. We can extend this basis to be a basis of $H^0(M, L^n)$, we denote such basis as $\{s_{n,1}^p, \dots, s_{n,d_n-1}^p, \Psi_n^q\}$. By relation (12) again, the covariance kernel for the conditional Gaussian measure $(H_q^0(M, L^n), d\gamma_{d_n-1}^q)$ is

$$(52) \quad \Pi_n^q(z, w) = \Pi_n(z, w) - \Psi_n^q(z) \overline{\Psi_n^q(w)}.$$

And hence,

$$(53) \quad D_n(z|q) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\Pi_n(z, z) - |\Psi_n^q(z)|^2|.$$

To prove Theorem 3, it is enough to find the expression of $|\Psi_n^q|^2$. In the following computations, we will discard the local frames $e^{\otimes n}$ and dz for simplicity. We write $s_{n,j}^q = f_{n,j}^q e^{\otimes n}$ locally. Then the Bergman kernel reads

$$\Pi_n(z, w) = \sum_{j=1}^{d_n-1} f_{n,j}^q(z) \overline{f_{n,j}^q(w)} + \Psi_n^q(z) \overline{\Psi_n^q(w)}.$$

We take the Chern connection $\nabla_{h^n}^z$ on both sides with respect to variable z and evaluate at $z = q$, we have the relation,

$$\nabla_{h^n}^z \Pi_n(z, w)|_{z=q} = \nabla_{h^n}^z \Psi_n^q(z)|_{z=q} \overline{\Psi_n^q(w)}.$$

This implies that $\Psi_n^q(w)$ is parallel to $\overline{\nabla_{h^n}^z \Pi_n(z, w)|_{z=q}}$. We define

$$\Psi_n^q(w) = \lambda_q \overline{\nabla_{h^n}^z \Pi_n(z, w)|_{z=q}}.$$

We will find $|\lambda_q|^2$ in order to get $|\Psi_n^q(w)|^2$ in (53). By definition of the Chern connection (7), we have

$$\Psi_n^q(w) = \lambda_q \left[\overline{\frac{\partial \Pi_n(z, w)}{\partial z}} \Big|_{z=q} - n \frac{\partial \phi}{\partial z}(q) \Pi_n(q, w) \right].$$

We can choose the Kähler normal coordinate freezing at the point $z_0 = q$ as the origin of the coordinate patch to simplify our computations. Recall equation (26), at the origin of the Kähler normal coordinate, we have $\frac{\partial \phi}{\partial z}(q) = 0$, and hence locally,

$$(54) \quad \Psi_n^q(w) = \lambda_q \overline{\frac{\partial \Pi_n(z, w)}{\partial z}} \Big|_{z=q}.$$

We can further rewrite $\Psi_n^q(w)$ as follows: choose any orthonormal basis $\{\psi_{n,1}, \dots, \psi_{n,d_n}\}$ of $H^0(M, L^n)$ with respect to the inner product (8) (or (9)), then the Bergman kernel is

$$\Pi_n(z, w) = \sum_{j=1}^{d_n} \psi_{n,j}(z) \overline{\psi_{n,j}(w)},$$

thus,

$$\Psi_n^q(w) = \lambda_q \sum_{j=1}^{d_n} \overline{\frac{\partial \psi_{n,j}}{\partial z}(q)} \psi_{n,j}(w).$$

Note that the L^2 -norm of $\Psi_n^q(w)$ is 1 by assumption, hence,

$$\begin{aligned} 1 &= \|\Psi_n^q(w)\|_{h^n}^2 = \left\| \lambda_q \sum_{j=1}^{d_n} \overline{\frac{\partial \psi_{n,j}}{\partial z}(q)} \psi_{n,j}(w) \right\|_{h^n}^2 \\ &= |\lambda_q|^2 \sum_{j=1}^{d_n} \frac{\partial \psi_{n,j}}{\partial z}(q) \overline{\frac{\partial \psi_{n,j}}{\partial z}(q)} = |\lambda_q|^2 (\partial_z \partial_{\bar{z}} \Pi_n)(q, q) \end{aligned}$$

Hence, combining the expression of $|\lambda_q|^2$ with (54), we have,

$$|\Psi_n^q(w)|^2 = \frac{|\partial_z \Pi_n(z, w)|_{z=q}|^2}{(\partial_z \partial_{\bar{z}} \Pi_n)(q, q)}.$$

Note that $\overline{\Pi_n(z, w)} = \Pi_n(w, z)$, thus $|\partial_z \Pi_n(z, w)|_{z=q}|^2 = |\partial_{\bar{z}} \overline{\Pi_n(z, w)}|_{z=q}|^2 = |\partial_{\bar{z}} \Pi_n(w, z)|_{z=q}|^2 = |\partial_{\bar{w}} \Pi_n(z, w)|_{w=q}|^2$, thus,

$$(55) \quad |\Psi_n^q(w)|^2 = \frac{|\partial_{\bar{w}} \Pi_n(z, w)|_{w=q}|^2}{(\partial_z \partial_{\bar{z}} \Pi_n)(q, q)}.$$

Now we complete the proof of Theorem 3 if we combine (53)(55).

8. PROOF OF THEOREM 4

8.1. Gaussian random $SU(2)$ polynomials. In this subsection, let's compute the rescaling limit $D_\infty(z|0)$ for Gaussian random $SU(2)$ polynomials.

We choose the affine coordinate at $q = 0$. As in §5, we still use the normalized Bergman kernel

$$\Pi_n(z, w) = (1 + z\bar{w})^n.$$

Thus we have,

$$(\partial_{\bar{w}}\Pi_n)(z, 0) = nz, \quad (\partial_z\partial_{\bar{z}}\Pi_n)(0, 0) = n.$$

Thus we have the following exact formula for the $SU(2)$ polynomials,

$$D_n^{SU(2)}(z|0) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log((1 + |z|^2)^n - n|z|^2).$$

We expand the right hand side to get,

$$\begin{aligned} D_n^{SU(2)}(z|0) &= \frac{1}{\pi} \left[\frac{n((1 + |z|^2)^{n-1} - 1 + (n-1)(1 + |z|^2)^{n-2}|z|^2)}{(1 + |z|^2)^n - n|z|^2} \right. \\ &\quad \left. - \frac{n^2((1 + |z|^2)^{n-1} - 1)|z|^2}{((1 + |z|^2)^n - n|z|^2)^2} \right] \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}. \end{aligned}$$

Now we rescale $z \rightarrow \frac{z}{\sqrt{n}}$ to get the limit,

$$\begin{aligned} D_\infty^{SU(2)}(z|0) &:= \lim_{n \rightarrow \infty} D_n^{SU(2)}(q + \frac{z}{\sqrt{n}}|q) \\ &= \frac{1}{\pi} \left[\frac{(e^{|z|^2} - 1 + e^{|z|^2}|z|^2)}{e^{|z|^2} - |z|^2} - \frac{(e^{|z|^2} - 1)^2|z|^2}{(e^{|z|^2} - |z|^2)^2} \right] \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}, \end{aligned}$$

which can be rewritten as

$$D_\infty^{SU(2)}(z|0) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(e^{|z|^2} - |z|^2).$$

This proves Theorem 4 for Gaussian random $SU(2)$ polynomials.

8.2. Proof of Theorem 4. Let's turn to prove Theorem 4 for the general cases.

We have to apply the similar estimates about the Bergman kernel as in §6. We continue to use the Kähler normal coordinate with the origin at $q = 0$ as in §7.

By Theorem 3, the rescaling limit of $D_n(z|q)$ is given as

$$D_n(\frac{v}{\sqrt{n}}|0) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left| \Pi_n(\frac{v}{\sqrt{n}}, \frac{v}{\sqrt{n}}) - \frac{|(\partial_{\bar{w}}\Pi_n)(\frac{v}{\sqrt{n}}, 0)|^2}{(\partial_z\partial_{\bar{z}}\Pi_n)(0, 0)} \right|.$$

Theorem 4 follows once we find the rescaling limits of $\Pi_n(\frac{v}{\sqrt{n}}, \frac{v}{\sqrt{n}})$ and $(\partial_{\bar{w}}\Pi_n)(\frac{v}{\sqrt{n}}, 0)$ and the estimate of $(\partial_z\partial_{\bar{z}}\Pi_n)(0, 0)$.

Let's recall the rescaling limit of the Bergman kernel off the diagonal,

$$\Pi_n(q + \frac{v}{\sqrt{n}}, q + \frac{u}{\sqrt{n}}) = n(e^{v \cdot \bar{u}} + \frac{1}{\sqrt{n}}p_1 + \dots).$$

As in §6, we first have the following asymptotics at $q = 0$,

$$(56) \quad \Pi_n(\frac{v}{\sqrt{n}}, \frac{v}{\sqrt{n}}) \sim ne^{|v|^2}, \quad (\partial_{\bar{w}}\Pi_n)(\frac{v}{\sqrt{n}}, 0) \sim n^{\frac{3}{2}}v.$$

Let's recall the C^∞ -expansion of the Bergman kernel on the diagonal,

$$\Pi_n(z, z) = ne^{n\phi}(1 + a_1(z)n^{-1} + a_2(z)n^{-2} + \cdots).$$

We take $\partial\bar{\partial}$ on both sides to get the full expansion,

$$\begin{aligned} (\partial_z \partial_{\bar{z}} \Pi_n)(z, z) &= n^3 e^{n\phi} |\partial\phi|^2 (1 + a_1 n^{-1} + \cdots) + n^2 e^{n\phi} \partial\bar{\partial}\phi (1 + a_1 n^{-1} + \cdots) \\ &\quad + 2n^2 e^{n\phi} \Re(\partial\phi(\partial\bar{a}_1 n^{-1} + \cdots)) + ne^{n\phi} (\partial\bar{\partial}a_1 n^{-1} + \cdots). \end{aligned}$$

Using identities (26) at the origin of the Kähler normal coordinate, we have,

$$(57) \quad (\partial_z \partial_{\bar{z}} \Pi_n)(0, 0) = n^2 + na_1 + (\partial\bar{\partial}a_1 + a_2) + O(n^{-1}).$$

If we combine the asymptotics (56)(57), we have the universal limit

$$D_\infty(v|0) := \lim_{n \rightarrow \infty} D_n\left(\frac{v}{\sqrt{n}}|0\right) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(e^{|v|^2} - |v|^2 \right),$$

which completes the proof of Theorem 4.

REFERENCES

- [1] R. Adler and J. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics Springer, New York (2007).
- [2] J. Baber, *Scaled correlations of critical points of Random sections on Riemann surfaces*, J Stat Phys (2012)148: 250–279.
- [3] P. Bleher, B. Shiffman and S. Zelditch, *Universality and scaling of correlations between zeros on complex manifolds*, Invent. Math. 142 (2000), 351–395.
- [4] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman Kernel*, J. Differential Geom. 72 (2006), no.1, 1–41.
- [5] M. R. Douglas, B. Shiffman and S. Zelditch, *Critical Points and Supersymmetric Vacua I*, Commun. Math. Phys. 252, 325–358 (2004).
- [6] M. R. Douglas, B. Shiffman and S. Zelditch, *Critical Points and Supersymmetric Vacua II: asymptotics and extremal metrics*, J. Differential. Geom. 72, (2006), 381–427.
- [7] G. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, (1978).
- [8] B. Hanin, *Pairing of Zeros and Critical Points for Random Meromorphic Functions on Riemann Surfaces*, Math. Res. Lett. 22 (2015), no 1, 111–140.
- [9] B. Hanin, *Correlations and Pairing Between Zeros and Critical Points of Gaussian Random Polynomials*, IMRN 2015, no. 2, 381–421.
- [10] J. H. Hannay, *Chaotic analytic zero points: exact statistics for those of a random spin state*, J. Phys. A 29 (1996), 314–320.
- [11] J. Hough, M. Krishnapur, Y. Peres and B. Virag, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, AMS, 2010.
- [12] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Math., Vol. 254, Birkhäuser, Basel, 2007.
- [13] B. Shiffman and S. Zelditch, *Number variance of random zeros on complex manifolds*, Geom.funct.anal. Vol. 18 (2008), 1422–1475.
- [14] B. Shiffman, S. Zelditch and Q. Zhong, *Random zeros on complex manifolds: conditional expectations*, Journal of the Institute of Mathematics of Jussieu, Volume 10, Special Issue 03, July 2011, pp 753–783.
- [15] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. 32 (1990), Math. Volume 13, Number 4 (1963), 1171–1180.
- [16] S. T. Yau, *Survey on partial differential equations in differential geometry. Seminar on Differential Geometry*, pp. 3–71, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.
- [17] S. Zelditch, *Szegő kernels and a theorem of Tian*, IMRN 6 (1998), 317–331.

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING, CHINA

E-mail address: renjie@math.pku.edu.cn